

# Einstein Static Universe in Exponential $f(T)$ Gravity

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## Abstract

We analyze the stability of the Einstein static closed and open universe in two types of exponential  $f(T)$  gravity theories. We show that the stable solutions exist in these two models. In particular, we find that large regions of parameter space in equation of state  $w = p/\rho$  for the stable universe are allowed in the  $f(T)$  theories.

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## I. INTRODUCTION

The Einstein static universe has recently been revived since our universe might evolve from it to inflation. This is called the emergent universe with an inflationary singularity to avoid a big bang singularity [1].

One way to look at the theory beyond general relativity (GR) is the teleparallel equivalence of general relativity (TEGR) [2–6] introduced first by Albert Einstein [7]. Different from GR with the Levi-Civita connection, teleparallel gravity (TG) uses the Weitzenbock connection, which has no curvature (R) but has torsion (T). To explain the late-time acceleration of the universe, a modified gravity has been proposed by extending T in the TG action to an arbitrary function  $f(T)$ . In  $f(T)$  gravity, torsion is responsible for the acceleration of the universe [8, 9]. This modified gravity would not only avoid the big bang singularity [10] but also provide an alternative to inflation [11]. However,  $f(T)$  gravity has some intrinsic problems, such as the violation of local Lorentz invariance [12]. In some of modified gravity models, the solutions of the stable Einstein static universe do exist under linear homogeneous scalar perturbations as shown in Refs. [13, 14]. In Ref. [15], the stability of the Einstein static universe in a power law  $f(T)$  plus the cosmological constant  $\Lambda$  is studied. However,  $\Lambda$  is at the pre-inflation scale, which may lead to the hierarchy problem in the current universe. To avoid this problem, we examine the stability of the Einstein static universe in two kinds of exponential  $f(T)$  gravity theories [9], which could explain the static Einstein universe in the pre-inflation era and be reduced back to TEGR in the present universe.

The paper is organized as follows. In Sec. II, we briefly introduce the formulation of  $f(T)$  gravity with the Weitzenbock connection. We also show two explicit forms of the exponential theories. In Sec. III, we give the conditions for the stable Einstein static solutions. In Sec. IV, we study the stable solutions in the exponential models. We illustrate the oscillating behavior of the Einstein universe in Sec. V. We present our conclusions in Sec. VI.

## II. FORMULATION IN $f(T)$ GRAVITY

In teleparallelism, the dynamical object is the vierbein field  $e_i(x^\mu)$ , which is an orthogonal basis for the tangent space at the point  $x^\mu$  of the manifold with the relation:  $e_i \cdot e_j = \eta_{ij}$ ,

where  $\eta_{ij} = \text{diag}(1, -1, -1, -1)$ . The metric tensor is given by

$$g_{\mu\nu} = \eta_{ij} e_\mu^i(x) e_\nu^j(x) \quad (2.1)$$

with

$$e_\mu^i(x) e_j^\mu(x) = \delta_j^i. \quad (2.2)$$

In this formulation, the Weitzenbock connection is used and the torsion tensor is defined by

$$T_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda - \Gamma_{\mu\nu}^\lambda = e_i^\lambda (\partial_\mu e_\nu^i - \partial_\nu e_\mu^i), \quad (2.3)$$

where

$$\Gamma_{\mu\nu}^\lambda = e_i^\lambda \partial_\nu e_\mu^i. \quad (2.4)$$

The action in TG is expressed as

$$I = \frac{1}{16\pi G} \int e T d^4x, \quad (2.5)$$

where  $e \equiv \det(e_\mu^i) = \sqrt{-g}$  and  $T$  is the torsion scalar, defined by

$$T = S_\lambda{}^{\mu\nu} T_{\mu\nu}^\lambda, \quad (2.6)$$

with

$$S_\lambda{}^{\mu\nu} \equiv \frac{1}{2} (K^{\mu\nu}{}_\lambda + \delta^\mu{}_\lambda T^{\theta\nu}{}_\theta - \delta^\nu{}_\lambda T^{\theta\mu}{}_\theta), \quad (2.7)$$

and

$$K^{\mu\nu}{}_\lambda = \frac{-1}{2} (T^{\mu\nu}{}_\lambda - T^{\nu\mu}{}_\lambda - T_\lambda{}^{\mu\nu}). \quad (2.8)$$

The modified teleparallel action for  $f(T)$  gravity is given by [9]

$$I = \frac{1}{16\pi G} \int e f(T) d^4x, \quad (2.9)$$

where  $f(T)$  is an arbitrary function of  $T$ . In this paper, we will concentrate on the following two types of the exponential  $f(T)$  models:

$$f(T) = T + \alpha T (1 - e^{\beta T_0/T}), \quad (2.10)$$

and

$$f(T) = T + \alpha T_0 \left( 1 - e^{\beta T^2/T_0^2} \right), \quad (2.11)$$

where  $\alpha$ ,  $\beta$  and  $T_0$  are constants. The motivation of these two models is that  $f(T)$  can be reduced back to TEGR for small  $T$ , corresponding to the current universe. Explicitly, when  $T$  is very small compared to  $T_0$  and  $\beta < 0$ , the two models in Eqs. (2.10) and (2.11) give  $f(T) \approx (1 + \alpha)T$  and  $f(T) \approx T$ , respectively.

### III. EINSTEIN STATIC UNIVERSE

We now consider the FRW metric

$$ds^2 = dt^2 - k^2 a^2(t) [d(k\psi)^2 + \sin^2(k\psi)(d\theta^2 + \sin^2\theta d\phi^2)] \quad (3.1)$$

where its vierbein fields are given by [16]

$$\begin{aligned} e_0^0 &= 1, \quad e_\psi^0 = e_\theta^0 = e_\phi^0 = e_0^1 = e_0^2 = e_0^3 = 0, \\ e_\psi^1 &= -a(t)k^2 \cos\theta, \quad e_\theta^1 = a(t)k \sin(k\psi) \sin\theta \cos(k\psi), \quad e_\phi^1 = -a(t)k \sin^2(k\psi) \sin^2\theta, \\ e_\psi^2 &= a(t)k^2 \sin\theta \cos\phi, \quad e_\theta^2 = -a(t)k \sin(k\psi) [\sin(k\psi) \sin\phi - \cos(k\psi) \cos\theta \cos\phi], \\ e_\phi^2 &= -a(t)k \sin(k\psi) \sin\theta [\cos(k\psi) \sin\phi + \sin(k\psi) \cos\theta \cos\phi], \\ e_\psi^3 &= -a(t)k^2 \sin\theta \sin\phi, \quad e_\theta^3 = -a(t)k \sin(k\psi) [\sin(k\psi) \cos\phi + \cos(k\psi) \cos\theta \sin\phi], \\ e_\phi^3 &= -a(t)k \sin(k\psi) \sin\theta [\cos(k\psi) \cos\phi - \sin(k\psi) \cos\theta \sin\phi], \end{aligned} \quad (3.2)$$

with  $k = 1$  and  $i$  or  $k^2 = \pm 1$ , representing the closed and open universe, respectively. The torsion scalar can be written as

$$T = 6(k^2 a^{-2} - H^2). \quad (3.3)$$

As a result, the modified Friedmann equation is given by [16]

$$12H^2 f'(T) + f(T) = 16\pi G\rho \equiv \kappa\rho, \quad (3.4)$$

$$(k^2 a^{-2} + \dot{H})(48H^2 f''(T) + 4f'(T)) - f'(T)(8\dot{H} + 12H^2) - f(T) = \kappa p. \quad (3.5)$$

To get an Einstein static universe, the conditions of  $\dot{a} = H = 0$ ,  $\ddot{a} = 0$ , and  $T_0 = T(a_0) = k^2 6/a_0^2$  are imposed. By using Eqs. (3.4) and (3.5), we obtain

$$f_0 \equiv f(T_0) = \kappa\rho_0, \quad (3.6)$$

$$k^2 \frac{4f'_0}{a_0^2} - f_0 = \kappa p_0, \quad (3.7)$$

with  $f'_0 \equiv df/dT|_{T=T_0}$ ,  $\rho_0 = \rho(a_0)$  and  $p_0 = p(a_0)$ . By combining Eqs. (3.6) and (3.7), we find

$$\left( \frac{Tf'}{f} \right)_{T=T_0} = \frac{3}{2} (1 + \mathbf{w}), \quad (3.8)$$

where  $\mathbf{w} = p/\rho$  is equation of state of the background matter.

We now perform the linear homogeneous scalar perturbations in the static Einstein universe. The perturbations in  $a$  and  $\rho$  depend only on time, *i.e.*,

$$a(t) = a_0(1 + \delta a(t)), \quad \rho(t) = \rho_0(1 + \delta \rho(t)), \quad (3.9)$$

with [15]

$$\delta a(t) = C_1 e^{\gamma t} + C_2 e^{-\gamma t}. \quad (3.10)$$

By using the procedure in Ref. [15], we get

$$\gamma^2 = \frac{\kappa \rho_0}{4 f_0'^2} (1 + w) \left[ (1 + 3w) f_0' - 3(1 + w) \frac{f_0 f_0''}{f_0'} \right]. \quad (3.11)$$

From Eq. (3.10), we obtain an oscillating universe if  $\gamma^2 < 0$ , corresponding to the stable Einstein static universe.

In the following section, we discuss the stable Einstein static solutions of the exponential  $f(T)$  models in Eqs. (2.10) and (2.11).

## IV. CONDITIONS FOR STABLE EINSTEIN STATIC SOLUTIONS

### A. The first type of the exponential models

In the closed universe, we have  $T_0 = 6a_0^{-2}$  for the Einstein static solution. Substituting Eq. (2.10) into Eqs. (3.6) and (3.7), we obtain the constraints for the Einstein static universe

$$T_0(1 + \alpha - \alpha e^\beta) = \kappa \rho_0, \quad (4.1)$$

$$\frac{2}{3} T_0 \alpha \beta e^\beta = \kappa \rho_0 \left( w + \frac{1}{3} \right), \quad (4.2)$$

leading to

$$\frac{T_0}{\kappa \rho_0} = 1 - \frac{1 - e^\beta}{2\beta e^\beta} (1 + 3w) > 0, \quad (4.3)$$

where  $T_0/(\kappa \rho_0)$  is positive since  $\rho_0$  is positive definite and  $T_0 > 0$  in the closed universe.

From Eqs. (2.10), (3.11), (4.1) and (4.2), we have

$$\gamma^2 = \frac{3\kappa^2 \rho_0^2}{8T_0 f_0'^2} (1 + w) (1 + 3w) \left[ \left( 1 + \frac{2\beta}{3} \right) + w \right]. \quad (4.4)$$

Note that  $f(T) \rightarrow T + \alpha \beta T_0$  when  $\beta \rightarrow 0$ . If the combination  $\alpha \beta T_0$  is finite, we may define it as the cosmological constant  $\Lambda$ . As expected, in the limit of  $f(T) \rightarrow T + \Lambda$ , one has

$$\gamma^2 = \frac{\kappa \rho_0}{4} (1 + w) (1 + 3w), \quad \Lambda = \frac{\kappa \rho_0}{2} (1 + 3w), \quad (4.5)$$

which is the result in GR. For the stable universe in GR, which requires  $\gamma^2 < 0$ , one finds

$$-1 < \mathbf{w} < -\frac{1}{3}. \quad (4.6)$$

In Fig. 1 (left panel), we show the stable regions ( $\gamma^2 < 0$ ) based on Eqs. (4.3) and (4.4). For  $\beta = 0$ , as seen from the figure, we obtain the same allowed region of  $-1 < \mathbf{w} < -1/3$  as that in Eq. (4.6).

In the case of the open universe, from Eqs. (3.6) and (3.7) we get the constraints for the Einstein open static universe as:

$$T_0(1 + \alpha - \alpha e^\beta) = \kappa \rho_0, \quad (4.7)$$

$$\frac{2}{3}T_0\alpha\beta e^\beta = \kappa\rho_0\left(\mathbf{w} + \frac{1}{3}\right), \quad (4.8)$$

respectively, where  $T_0 = -6a_0^{-2}$ . With these two constraints, we find

$$\frac{T_0}{\kappa\rho_0} = 1 - \frac{1 - e^\beta}{2\beta e^\beta} (1 + 3\mathbf{w}), \quad (4.9)$$

which is negative because  $T_0 = -6a_0^{-2} < 0$  in the static open universe. From  $f(T)$  in Eq. (3.11), we derive

$$\gamma^2 = \frac{3\kappa^2\rho_0^2}{8T_0f_0'^2}(1 + \mathbf{w})(1 + 3\mathbf{w})\left(1 + \frac{2\beta}{3} + \mathbf{w}\right). \quad (4.10)$$

In Fig. 1 (right panel), we display the stable solutions ( $\gamma^2 < 0$ ) for the Einstein static open universe. Clearly, there is no stable region at  $\beta = 0$ . Thus, the universe is not stable in the spatially open universe.

## B. The second Type of the exponential models

For the second type of the exponential models in Eq. (2.11), we obtain conditions for the closed Einstein static universe as

$$f_0 = T_0(1 + \alpha - \alpha e^\beta) = \kappa\rho_0, \quad (4.11)$$

$$\frac{2}{3}T_0(1 - 2\alpha\beta e^\beta) - \kappa\rho_0 = \kappa p_0, \quad (4.12)$$

leading to

$$\frac{T_0}{\kappa\rho_0} = \frac{\frac{3}{2}(1 + \mathbf{w})(1 - e^\beta) + 2\beta e^\beta}{1 - e^\beta + 2\beta e^\beta}, \quad (4.13)$$

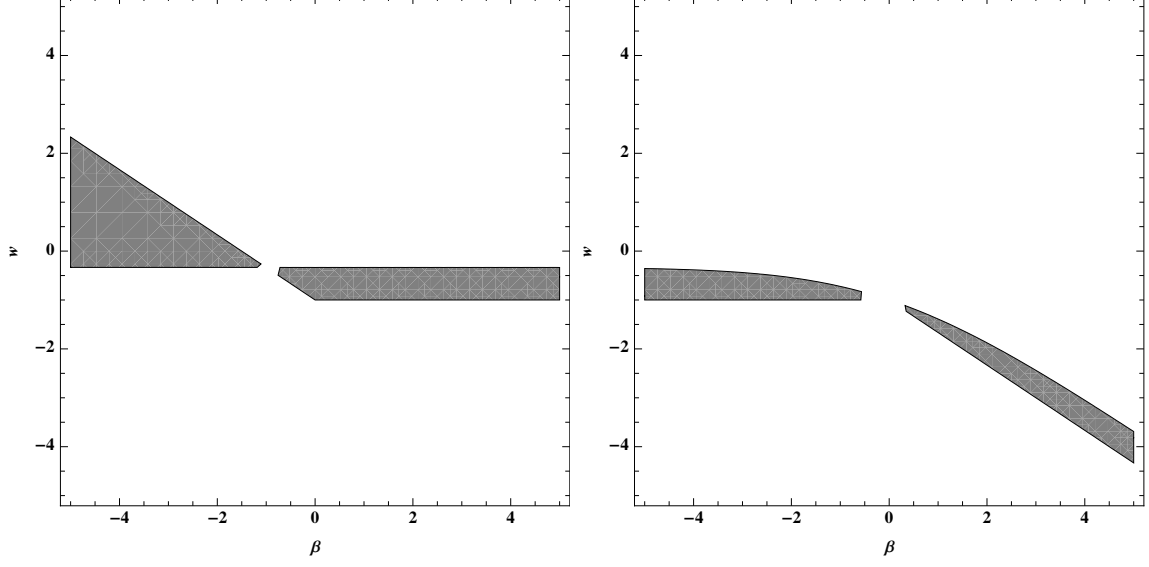


FIG. 1. Stable solutions (shaded regions) of the first exponential gravity model in the spatially closed (left panel) and open (right panel) Einstein static universe.

which will be positive because  $T_0 = 6a_0^{-2} > 0$  in the closed universe. Substituting Eq. (2.11) into Eq. (3.11), we get

$$\gamma^2 = \frac{3\kappa^2\rho_0^2}{8T_0f_0'^2}(1+w) \left[ (1+w)(1+3w) - 2(1+2\beta) \left( w+1 - \frac{(1+w)(1-e^\beta) + \frac{4}{3}\beta e^\beta}{1-e^\beta + 2\beta e^\beta} \right) \right]. \quad (4.14)$$

From Eqs. (4.13) and (4.14), we can plot the stable solutions ( $\gamma^2 < 0$ ) of the Einstein static closed universe in Fig. 2 (left panel).

In the open universe, the forms of equations are the same as those in the closed universe. However, since  $T_0 = -6a_0^{-2} < 0$  in the static open universe, the constraint in Eq. (4.15) is negative, *i.e.*

$$\frac{T_0}{\kappa\rho_0} = \frac{\frac{3}{2}(1+w)(1-e^\beta) + 2\beta e^\beta}{1-e^\beta + 2\beta e^\beta} < 0, \quad (4.15)$$

while  $\gamma^2$  is also negative for the stable static universe:

$$\gamma^2 = \frac{3\kappa^2\rho_0^2}{8T_0f_0'^2}(1+w) \left[ (1+w)(1+3w) - 2(1+2\beta) \left( w+1 - \frac{(1+w)(1-e^\beta) + \frac{4}{3}\beta e^\beta}{1-e^\beta + 2\beta e^\beta} \right) \right]. \quad (4.16)$$

The stable solutions for the open universe are illustrated in Fig. 2 (right panel).

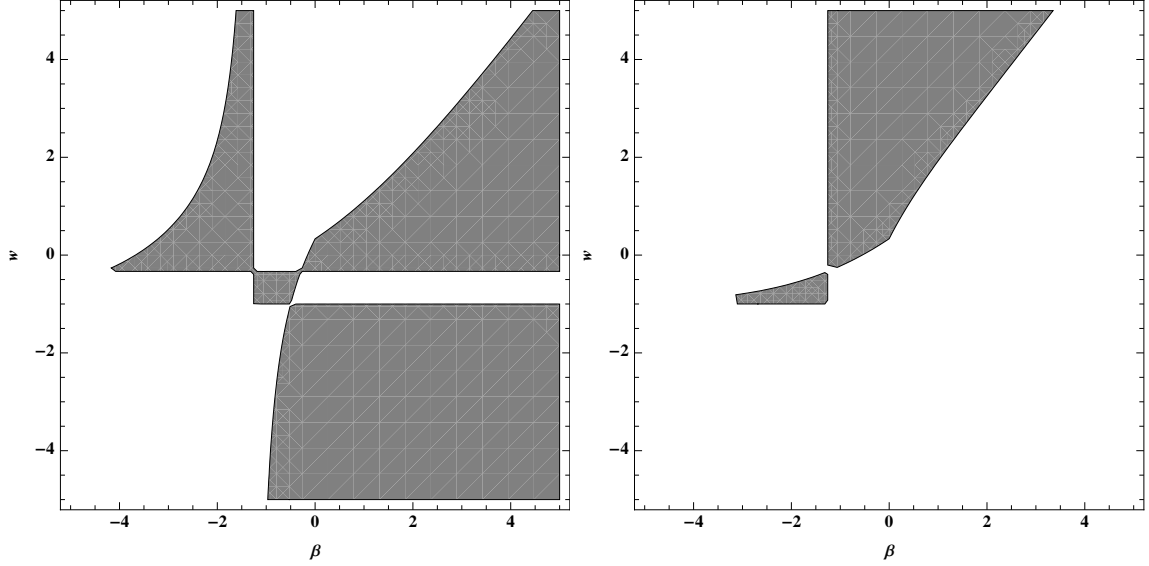


FIG. 2. Legend is the same as Fig. 1 but for the second model.

## V. OSCILLATING EINSTEIN UNIVERSE

From Eq. (3.10), it is obvious that the stable Einstein static universe is oscillating. The behavior has also been explicitly demonstrated in other gravity theories, such as the vacuum energy model [17] and DGP braneworld scenario [14]. In this section, we would like to give similar discussions to illustrate this oscillating behavior in the first type of the exponential models in Eq. (2.10). The results can be easily extended to the second one.

Since an obvious stable solution for the first exponential closed universe comes from  $\beta = -2$  and  $w = 0$ , we can get

$$\gamma = \sqrt{\frac{3\kappa^2\rho_0^2}{8T_0f_0'^2}} \cdot \frac{i}{\sqrt{3}} \quad (5.1)$$

from Eq. (4.4). Consequently, the scale factor is given by

$$a(t) = a_0 \left[ 1 + C \sin \left( \sqrt{\frac{3\kappa^2\rho_0^2}{8T_0f_0'^2}} \frac{t}{\sqrt{3}} + \eta \right) \right]. \quad (5.2)$$

By defining  $t' \equiv \sqrt{\frac{3\kappa^2\rho_0^2}{8T_0f_0'^2}} t$ , we get

$$\begin{aligned} a(t') &= a_0 \left[ 1 + C \sin \left( \frac{t'}{\sqrt{3}} + \eta \right) \right], \\ \dot{a}(t') &= \frac{a_0 C}{\sqrt{3}} \cos \left( \frac{t'}{\sqrt{3}} + \eta \right). \end{aligned} \quad (5.3)$$



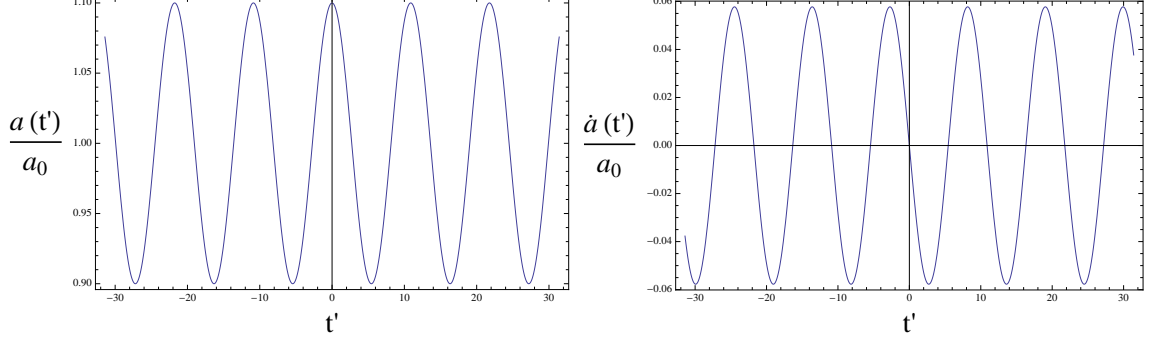


FIG. 3.  $a(t')$  (left panel) and  $\dot{a}(t')$  (right panel) as functions of  $t' (= \sqrt{\frac{3\kappa^2 \rho_0^2}{8T_0 f_0'^2}} t)$  in the first exponential gravity model in the closed universe.

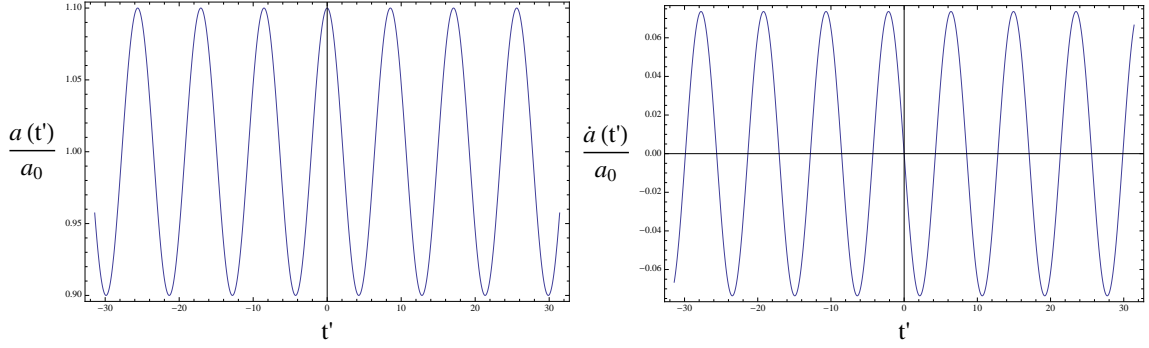


FIG. 4. Legend is the same as Fig. 3 but in the open universe with  $t' = \sqrt{\frac{3\kappa^2 \rho_0^2}{8|T_0| f_0'^2}} t$ .

Using initial conditions given by  $a(t=0) = 1.1 \times a_0$  and  $\dot{a}(t=0) = 0$ , we obtain  $\eta = \pi/2$  and  $C = 0.1$ . We plot the stable small oscillating closed universe for the first exponential gravity model in Fig. 3.

For the open universe, we pick the stable solution of  $\beta = -4$  and  $w = -1/2$ . With the same initial conditions as the closed universe, we find

$$\begin{aligned} \frac{a(t')}{a_0} &= \left[ 1 + \frac{1}{10} \sin \left( \sqrt{\frac{13}{24}} t' + \frac{\pi}{2} \right) \right], \\ \frac{\dot{a}(t')}{a_0} &= \frac{1}{10} \sqrt{\frac{13}{24}} \cos \left( \sqrt{\frac{13}{24}} t' + \frac{\pi}{2} \right), \end{aligned} \quad (5.4)$$

where  $t' = \sqrt{\frac{3\kappa^2 \rho_0^2}{8|T_0| f_0'^2}} t$ . In Fig. 4, we show the oscillation in the open case, which is not allowed in GR.

## VI. CONCLUSIONS

We have discussed the linear homogeneous scalar perturbations near the Einstein static universe in  $f(T)$  gravity. We have explicitly studied both closed and open universe in the two exponential  $f(T)$  gravity models. These two  $f(T)$  models are proposed in order to explain the pre-inflation universe and be reduced back to TEGR in the present universe. We have demonstrated that in these models, the Einstein static universe can be stable in both open and closed cases with large allowed regions of  $w$ . Note that in GR, only the closed Einstein universe contains stable solutions with  $-1 < w < -\frac{1}{3}$  for the background matter. Explicitly, for the first exponential gravity model of  $f(T) = T + \alpha T(1 - e^{\beta T_0/T})$ , we have shown that  $w > -1$  ( $w < -\frac{1}{3}$ ) in the closed (open) universe can have the stable solution. In the limit  $\beta \rightarrow 0$ , the model gives the GR result as  $f(T) \rightarrow T + \Lambda$  with  $\Lambda = \alpha\beta T_0$  being the cosmological constant. For the second exponential gravity model of  $f(T) = T + \alpha T_0(1 - e^{\beta T^2/T_0^2})$ , we have found that  $w > -1$  (arbitrary  $w$ ) is the stable solution for the open (closed) Einstein universe. We have also illustrated the oscillating behaviors of the stable Einstein static universe in the first model. We have shown that both  $a(t)$  and  $\dot{a}(t)$  are oscillating without divergences as expected.

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